PERTURBATIONS OF VON NEUMANN ALGEBRAS I STABILITY OF TYPE.

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1. **Introduction.** In this article we initiate the study of "perturbation" of operator algebras. We deal with the question of structural similarities in "neighboring" algebras and prove that "type" is preserved under small perturbations of a von Neumann alegbra (Theorem A). We show that suitably "close" von Neumann algebras ("close" in a sense to be made precise in the next section, Definition A, but, roughly, that the unit ball of each algebra is norm close to the unit ball of the other) have central projections corresponding to the various pure types which are close to one another. One concludes, for example, that a von Neumann algebra "near" a factor of type II_1 *is* a factor of type II_1.

On the path to these results, we draw several auxiliary results which have independent interest: neighboring von Neumann algebras have neighboring centers, minimal projections, abelian projections, etc.

In what sense can one speak of "perturbation" of a von Neumann algebra? We have not "moved" it by some process—"adding a term," for instance. There is such a process available, however. If the von Neumann algebra is transformed by a unitary operator close to the identity operator the result is a "slight perturbation" of the original algebra. In this case, the perturbed algebra retains, of course, all the structural features of the original. We believe that this is the *only* possible method of perturbing an operator algebra—that suitably close operator algebras are unitarily equivalent. The results of this paper may be viewed as a first step in a program to establish that unitary equivalence.

There are several other aspects of this perturbation theory which deserve study. Are "neighboring" representations (with the meaning evident from the direction of this paper) of an operator algebra equivalent? We feel that they are. Are they "inner" equivalent? (Can the unitary equivalence be effected by a unitary operator in the von Neumann algebra generated by the

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images of the representations?) With somewhat less conviction, we feel that this, too, is the case. The result of [5; Theorem 7] that an automorphism α closer to the identity automorphism than 2 is inner, appears as corroborative evidence.

This work has its origins in the considerations surrounding nets of algebras [4] and, in a more primitive sense, in Glimm's uhf algebras [3] and the "approximately finite" algebras [9; Chapter IV] of Murray and von Neumann. The possibility of transforming one net for an algebra onto another raises questions similar to those dealt with in this paper.

The main results are found in the third and fourth sections. In the next section, we introduce the basic "measurement" definition and conduct some preliminary studies of the estimates which will appear in the later proofs.

2. Notation and definitions. If $\mathcal{F}$ denotes a family of bounded operators on the Hilbert space $\mathcal{H}$, we use the (standard) notation $\mathcal{F}'$ to designate the family of bounded operators on $\mathcal{H}$ commuting with $\mathcal{F}$ (the *commutant* of $\mathcal{F}$), and the notation $\mathcal{F}_1$ for the set of operators in $\mathcal{F}$ having bound not exceeding 1. The algebra of all bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. For each bounded operator A, $\| A - \mathcal{F} \| = \inf\{\|A - F\| : F \text{ in } \mathcal{F}\}$. The The strong-operator closure of the subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{S}^-$.

Definition A. If $\mathcal{A}$ and $\mathcal{B}$ are linear subspaces of $\mathcal{B}(\mathcal{H})$,

$$\| \mathcal{A} - \mathcal{B} \| = \sup\{\|A - \mathcal{B}_1\|, \| B - \mathcal{A}_1 \| : A \text{ in } \mathcal{A}_1, B \text{ in } \mathcal{B}_1\}.$$

If $\| \mathcal{A} - \mathcal{B} \| < a$ and S is in the unit ball of either $\mathcal{A}$ or $\mathcal{B}$, there is a T in the unit ball of the other such that $\| S - T \| < a$.

In connection with Lemma 2, our estimates, throughout the paper, will involve $a + \frac{1}{2} - (\frac{1}{4} - 2a)^{\frac{1}{2}}$ when the intial assumptions involve a. We denote this expression by $\alpha(a)$ in all that follows. Our interest is in knowing that $\alpha(a)$ is dominated by certain numbers provided a is (positive, and) dominated by other numbers. Note that α is defined on $[0, \frac{1}{8}]$, monotone increasing there, and has values in $[0, \frac{5}{8}]$. For the reader's convenience, we describe the simple considerations which lead to estimates we shall use (without additional comment). If $n \geqq 8$, then

$$\alpha(\frac{1}{n}) = (2n)^{-1}[n + 2 - ((n-4)^2 - 16)^{\frac{1}{2}}].$$

Now $(n-4)^2 - 16 \geqq [(n-4) - 2t]^2$ when $n - 4 \geqq 4t^{-1} + t$. For such n, $\alpha(\frac{1}{n}) \leqq (3 + t)n^{-1}$. In any event, with $n \geqq 9$, $\alpha(\frac{1}{n}) \leqq \frac{4}{n}$. For large n, $\alpha(\frac{1}{n})$ is almost majorized by $\frac{3}{n}$.

3. Structural estimates. The lemmas which follow allow us to estimate the degree to which structural elements (minimal projections, central projections, etc.) of an operator algebra can be approximated by elements having similar structural properties relative to a neighboring algebra.

LEMMA 1. *If $\mathfrak{A}$ is a self-adjoint algebra of operators acting on the Hilbert space $\mathcal{H}$, $0 < c < 1$, and F is a non-zero projection in $\mathfrak{A}^-$ such that for each self-adjoint A in the unit ball of $\mathfrak{A}$ there is a scalar b for which $\| FAF - bF \| \leqq c$; then F is a minimal projection in $\mathfrak{A}^-$.*

Proof. Suppose $F = M + N$, with M and N non-zero projections in $\mathfrak{A}^-$. Using the Kaplansky Density Theorem [6, 1; Théorème 3, p. 43], there is a self-adjoint A in the unit ball of $\mathfrak{A}$ such that $\| [A - (M - N)]x_0 \| < \epsilon$ and $\| [A - (M - N)]y_0 \| < \epsilon$, where x_0, a unit vector in $M(\mathcal{H})$, y_0, a unit vector in $N(\mathcal{H})$, and ϵ (> 0) are preassigned. Then

$$\| Ax_0 - x_0 \| < \epsilon \text{ and } \| Ay_0 + y_0 \| < \epsilon$$

By assumption, there is a scalar b such that $\| FAF - bF \| \leqq c$. Thus,

$$- \| FAx_0 - x_0 \| + \| x_0 - bx_0 \| \leqq \| FAx_0 - bx_0 \| \leqq \| FAF - bF \| \leqq c;$$

and

$$1 - b \leqq c + \| FAx_0 - x_0 \| = c + \| F(Ax_0 - x_0) \| < c + \epsilon.$$

At the same time,

$$\| by_0 + y_0 \| - \| y_0 + FAy_0 \| \leqq \| FAy_0 - by_0 \| \leqq \| FAF - bF \| \leqq c$$

and

$$1 + b \leqq c + \| FAy_0 + y_0 \| < c + \epsilon.$$

Thus $1 < c + \epsilon$. Choosing $\epsilon < 1 - c$, at the outset, we arrive at a contradiction. Hence no decomposition of the form $F = M + N$ is possible in $\mathfrak{A}^-$, and F is minimal in $\mathfrak{A}^-$.

LEMMA 2. *If $\mathfrak{A}$ is a C^*-algebra acting on the Hilbert space $\mathcal{H}$ and E is a projection on $\mathcal{H}$ such that $\| E - A \| < a$ $(\leqq \frac{1}{8})$ for some A in $\mathfrak{A}_1$, then $\| E - F \| < \alpha(a)$ for some projection F in $\mathfrak{A}$.*

Proof. Since $\| E - A \| < a$, $\| E - A^* \| < a$; and $\| E - \frac{1}{2}(A + A^*) \| < a$. We assume that A is self-adjoint. Note that,

$$\begin{aligned} \| A^2 - A \| &= \| A(I - A) - E(I - E) \| \\ &\leqq \| A(I - A) - A(I - E) \| + \| A(I - E) - E(I - E) \| \\ &< 2a. \end{aligned}$$

Passing to the function algebra representation of the C^*-algebra generated by A, we see that each point s of the spectrum, $\sigma(A)$, of A satisfies, $0 < s^2 - s + 2a$. Thus s does not lie in the interval

$$[\tfrac{1}{2} - (\tfrac{1}{4} - 2a)^{\frac{1}{2}}, \tfrac{1}{2} + (\tfrac{1}{4} - 2a)^{\frac{1}{2}}].$$

The function f defined as 0 to the left and 1 to the right of this interval is continuous on $\sigma(A)$; and $f(A)$ is a projection, F, in $\mathfrak{A}$. At the same time, $s - s^2 + 2a > 0$ for each s in $\sigma(A)$; so that s lies in

$$[\tfrac{1}{2} - (\tfrac{1}{4} + 2a)^{\frac{1}{2}}, \tfrac{1}{2} + (\tfrac{1}{4} + 2a)^{\frac{1}{2}}].$$

Thus

$$\| F - A \| \leqq \max\{ (\tfrac{1}{4} + 2a)^{\frac{1}{2}} - \tfrac{1}{2}, \tfrac{1}{2} - (\tfrac{1}{4} - 2a)^{\frac{1}{2}}\} = \tfrac{1}{2} - (\tfrac{1}{4} - 2a)^{\frac{1}{2}};$$

and

$$\| F - E \| \leqq \| F - A \| + \| A - E \| < \alpha(a).$$

LEMMA 3. *If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras acting on the Hilbert space $\mathcal{H}$, such that $\| \mathfrak{A} - \mathfrak{B} \| < a$ $(\leqq \frac{1}{16})$; then, with E a projection in $\mathfrak{A}$ minimal in $\mathfrak{A}^-$, there is a projection F in $\mathfrak{B}$ minimal in $\mathfrak{B}^-$ such that $\| E - F \| < \alpha(a)$. If P is a unit for $\mathfrak{A}$ and $\mathfrak{B}$, we can choose F so that $\| E - F \| < \alpha(\frac{1}{2}a)$.*

Proof. From Lemma 2, there is a projection F in $\mathfrak{B}$ such that $\| E - F \| < \alpha(a)$ $(\leqq \alpha(\frac{1}{16}) < \frac{1}{4})$. With B in $\mathfrak{B}_1$, choose A in $\mathfrak{A}_1$ such that $\| A - B \| < \frac{1}{8}$. For some scalar c, $EAE = cE$, since E is minimal in $\mathfrak{A}^-$. Thus

$$\| FBF - cF \| \leqq \| FBF - FAF \| + \| FAF - FAE \| + \| FAE - EAE \| + \| EAE - cE \| + \| cE - cF \| \leqq \tfrac{1}{8} + \tfrac{3}{4} < 1.$$

From Lemma 1, F is minimal in $\mathfrak{B}^-$.

If P is a unit for $\mathfrak{B}$, as well as for $\mathfrak{A}$, we may assume that $\| \mathfrak{A} - \mathfrak{B} \| < a$ $(\leqq \frac{1}{8})$ and conclude that $\| E - F \| < \alpha(\frac{1}{2}a)$. In this case, there is a T in $\mathfrak{B}_1$ such that $\| 2E - P - T \| < a$ $(\leqq \frac{1}{8})$; so that $\| E - \frac{1}{2}(P + T) \| < \frac{1}{2}a$ $(\leqq \frac{1}{16})$, with $\frac{1}{2}(P + T)$ in $\mathfrak{B}_1$. As before, there is a projection F in $\mathfrak{B}$ such that $\| E - F \| < \alpha(\frac{1}{2}a)$ $(\leqq \alpha(\frac{1}{16}) < \frac{1}{4})$, and the remainder of the argument applies as it stands.

LEMMA 4. *A projection E in a von Neumann algebra $\mathcal{R}$ is central if and only if $\| EA - AE \| < 1$ for each A in $\mathcal{R}_1$.*

Proof. Restricting to the range of the unit element of $\mathcal{R}$, we may assume $I \in \mathcal{R}$. Supposing E is not central, there is no central projection P such that $E \leqq P$ and $I - E \leqq I - P$. From the Comparison Theorem

[1; Lemme 1, p. 217], E and $I - E$ have subprojections E_0 and F_0, respectively, equivalent in $\mathcal{R}$. Let V be a partial isometry in $\mathcal{R}$ with initial space E_0 and final space F_0. Then

$$1 = \| V \| = \| EF_0V - VE_0E \| = \| EV - VE \|,$$

completing the proof.

Remark A. If $\mathcal{R}$ in Lemma 4 is replaced by a C^*-algebra $\mathfrak{A}$ and we assume that there is a constant c such that $\| EA - AE \| \leqq c < 1$ for each A in $\mathfrak{A}_1$, the Kaplansky Density Theorem and Lemma 4 allows us to conclude that E is central in $\mathfrak{A}$.

LEMMA 5. *If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras acting on a Hilbert space $\mathcal{H}$ and $\| \mathfrak{A} - \mathfrak{B} \| < a$ then $\| \mathfrak{A}^- - \mathfrak{B}^- \| < a$.*

Proof. With A_0 in $\mathfrak{A}_1^-$, b chosen such that $\| \mathfrak{A} - \mathfrak{B} \| < b < a$, and x, y unit vectors in in $\mathcal{H}$; the set $\mathcal{S}_{x,y}$ of operators B in $\mathfrak{B}_1^-$ such that $|([A_0 - B]x, y)| \leqq b$ is weak-operator closed in $\mathfrak{B}_1^-$. Given unit vectors $x_1, \cdots, x_n; y_1, \cdots, y_n$ in $\mathcal{H}$, choose c such that $\| \mathfrak{A} - \mathfrak{B} \| < c < b$ and A in $\mathfrak{A}_1$ (using the Kaplansky Density Theorem) such that $|([A_0 - A]x_j, y_j)| < b - c$, $j = 1, \cdots, n$. By choice of c, there is a B in $\mathfrak{B}_1$ (hence in $\mathfrak{B}_1^-$) such that $\| A - B \| < c$; so that $|([A - B]x_j, y_j)| < c$, $j = 1, \cdots, n$. Then $|([A_0 - B]x_j, y_j)| < b$ and $B \in \mathcal{S}_{x_j,y_j}$, $j = 1, \cdots, n$. Thus $\{\mathcal{S}_{x,y}\}$ has the finite intersection property. As $\mathfrak{B}_1^-$ is weak-operator compact (and the $\mathcal{S}_{x,y}$ are weak-operator closed subsets of $\mathfrak{B}_1^-$), there is a B_0 in all $\mathcal{S}_{x,y}$. Since $|([A_0 - B_0]x, y)| \leqq b$ for all unit vectors x, y in $\mathcal{H}$, $\| A_0 - B_0 \| \leqq b$. Thus $\| \mathfrak{A}^- - \mathfrak{B}^- \| < a$.

LEMMA 6. *If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras acting on a Hilbert space, $\| \mathfrak{A} - \mathfrak{B} \| < a$ $(\leqq \frac{1}{10})$, and P is a central projection in $\mathfrak{A}^-$, there is a central projection Q in $\mathfrak{B}^-$ such that $\| P - Q \| < \alpha(a)$. In particular, if P is not a scalar Q is not a scalar.*

Proof. From Lemma 2, there is a projection Q in $\mathfrak{B}^-$ such that $\| P - Q \| < \alpha(a)$ $(\leqq \alpha(\frac{1}{10}) < \frac{2}{5})$. With B in $\mathfrak{B}_1^-$, choose A in $\mathfrak{A}_1^-$ (using Lemma 5) such that $\| A - B \| < \frac{1}{10}$. Then

$$\begin{aligned} \| QB - BQ \| &\leqq \| QB - QA \| + \| QA - PA \| + \| PA - AQ \| + \| AQ - BQ \| \\ &\leqq 2(\| A - B \| + \| P - Q \|) < 1. \end{aligned}$$

From Lemma 4, Q is central in $\mathcal{B}^-$. Since $\| P - Q \| < 1$, if P is not a scalar, Q is not a scalar.

Remark B. Assuming P is central in $\mathcal{A}$, in the preceding lemma, there is a projection Q in $\mathcal{B}$, from Lemma 2, such that $\| P - Q \| < \alpha(a)$. As $\| \mathcal{A}^- - \mathcal{B}^- \| < a$ (Lemma 5), and P is central in $\mathcal{A}^-$, the argument of Lemma 6 shows that Q is central in $\mathcal{B}^-$ and, *a fortiori* in $\mathcal{B}$.

COROLLARY A. *If $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras acting on a Hilbert space and $\| \mathcal{M} - \mathcal{N} \| < \frac{1}{10}$, then $\mathcal{M}$ is a factor if and only if $\mathcal{N}$ is a factor.*

The argument of Lemma 6 yields, as well, the following.

COROLLARY B. *If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras acting on a Hilbert space, $\| \mathcal{A} - \mathcal{B} \| < \frac{1}{2}a$, E and F are projections in $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\| E - F \| < \frac{1}{2}(1 - a)$, then F is central if E is central.*

Proof. Choose b so that $\| E - F \| < \frac{1}{2}b < \frac{1}{2}(1 - a)$. From Lemma 5 and the proof of Lemma 6,

$$\| FB - BF \| \leqq 2(\| A - B \| + \| E - F \|) < a + b < 1,$$

for each B in $\mathcal{B}_1^-$, assuming E is central in $\mathcal{A}$. Hence, from Lemma 4, F is central in $\mathcal{B}$.

COROLLARY C. *If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras acting on a Hilbert space and $\| \mathcal{A} - \mathcal{B} \| < \frac{1}{10}$, then $\mathcal{A}$ is abelian if and only if $\mathcal{B}$ is abelian.*

COROLLARY D. *If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras acting on a Hilbert space, $\| \mathcal{A} - \mathcal{B} \| < \frac{1}{30}$, E and F are projections in $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\| E - F \| < \frac{1}{30}$; then E is an abelian projection in $\mathcal{A}^-$ if and only if F is an abelian projection in $\mathcal{B}^-$.*

Proof. With EAE in $(E\mathcal{A}E)_1$ (hence in $\mathcal{A}_1$), choose B in $\mathcal{B}_1$ such that $\| EAE - B \| < \frac{1}{30}$. Then $\| EAE - EBE \| = \| E(EAE - B)E \| < \frac{1}{30}$; and $\| EAE - FBF \| < \frac{1}{10}$. From Corollary C, $F\mathcal{B}F$ is abelian if and only if $E\mathcal{A}E$ is abelian. But E is an abelian projection in $\mathcal{A}^-$ if and only if $E\mathcal{A}E$ is abelian.

LEMMA 7. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* C^**-algebras acting on a Hilbert space,* $\| \mathfrak{A} - \mathfrak{B} \| < a$ $(\leqq \frac{1}{12})$, *E and F are projections in* $\mathfrak{A}$ *and* $\mathfrak{B}$, *respectively, and* $\| E - F \| < \alpha(a)$, *then* $\| C_E - C_F \| < 2\alpha(a)$, *where* C_E *and* C_F *are the central carriers of E and F relative to* $\mathfrak{A}^-$ *and* $\mathfrak{B}^-$.

Proof. From Lemma 5, $\| \mathfrak{A}^- - \mathfrak{B}^- \| < a$; so that we may assume that $\mathfrak{A}$ and $\mathfrak{B}$ are von Neumann algebras. By Lemma 6, there is a central projection Q in $\mathfrak{B}$ such that $\| C_E - Q \| < \alpha(a)$. Thus

$$\| F - QF \| \leqq \| F - E \| + \| C_E E - QE \| + \| QE - QF \| < 3\alpha(a) < 1.$$

Since QF is a subprojection of F; $F = QF$, and $C_F \leqq Q$.

Symmetrically, there is a central projection P in $\mathfrak{A}$ such that $\| C_F - P \| < \alpha(a)$ and $C_E \leqq P$. Thus

$$\| C_E(C_F - P) \| = \| C_E C_F - C_E \| < \alpha(a),$$
$$\| (C_E - Q) C_F \| = \| C_E C_F - C_F \| < \alpha(a);$$

so that $\| C_E - C_F \| < 2\alpha(a)$.

LEMMA 8. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* C^**-algebras acting on a Hilbert space,* $\| \mathfrak{A} - \mathfrak{B} \| < \frac{1}{8}$, *E, F* and *M, N are pairs of projections in* $\mathfrak{A}$ *and* $\mathfrak{B}$, *respectively,* $\| E - M \| < \frac{1}{8}$, *and* $\| F - N \| < \frac{1}{8}$, *then E is equivalent to F relative to* $\mathfrak{A}^-$ *if and only if M is equivalent to N relative to* $\mathfrak{B}^-$.

Proof. Suppose E is equivalent to F and V is a partial isometry in $\mathfrak{A}^-$ with initial space E and final space F. Choose B_0 in $\mathfrak{B}_1^-$ such that $\| V - B_0 \| < \frac{1}{8}$. Then, with $B = NB_0M$,

$$\| B - V \| = \| NB_0M - FVE \| < \tfrac{3}{8},$$
$$\| B^*B - E \| = \| B^*B - V^*V \| < \tfrac{3}{4},$$

and

$$\| B^*B - M \| \leqq \| B^*B - E \| + \| E - M \| < 1,$$

As B^*B lies in the Banach algebra $M\mathfrak{B}^-M$ with unit M; B^*B is invertible in that algebra. In particular, B^*B, hence B^*, has range that of M. Similarly, B has range that of N; and, from [8; Lemma 6.2.1], N is equivalent to M relative to $\mathfrak{B}^-$.

LEMMA 9. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* C^**-algebras acting on a Hilbert space,* $\| \mathfrak{A} - \mathfrak{B} \| < a$, *E and F are projections in* $\mathfrak{A}$ *and* $\mathfrak{B}$, *respectively,* $\| E - F \| < b$, *and* $a + 2b < \frac{1}{26}$; *then E is infinite relative to* $\mathfrak{A}^-$ *if and only if F is infinite relative to* $\mathfrak{B}^-$.

Proof. Suppose E is infinite in $\mathfrak{A}^-$. There is a proper subprojection E_0 of E in $\mathfrak{A}^-$ equivalent to E. As

$$\| E\mathfrak{A}^-E - F\mathfrak{B}^-F \| \leqq \| \mathfrak{A} - \mathfrak{B} \| + 2\| E - F \| < \frac{1}{26},$$

there is, by Lemma 2, a projection F_0 in $F\mathfrak{B}^-F$ such that $\| E_0 - F_0 \| < \alpha(\frac{1}{26}) < \frac{1}{8}$. From Lemma 8, F_0 is equivalent to F. Now $F_0 < F$ and F is infinite; for

$$\begin{aligned} \| F - F_0 \| &= \| E - E_0 - (E - F + F_0 - E_0) \| \\ &\geqq \| E - E_0 \| - (\| E - F \| + \| F_0 - E_0 \|) > 1 - \tfrac{1}{4} > 0. \end{aligned}$$

4. Stability results. In this section, we give estimates of the proximity of the various central portions corresponding to pure type of neighboring von Neumann algebras. These are, then, assembled in Theorem A, the main result.

LEMMA 10. *If $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras acting on a Hilbert space, $\| \mathcal{R} - \mathcal{S} \| < a$ $(\leqq \frac{1}{92})$ and P_d, Q_d are maximal central projections in $\mathcal{R}$, $\mathcal{S}$, respectively, such that $\mathcal{R}P_d$ and $\mathcal{S}Q_d$ are of type I; then $\| P_d - Q_d \| < 2\alpha(a)$.*

Proof. Since $\mathcal{R}P_d$ is of type I, there is an abelian projection E in $\mathcal{R}P_d$ with $C_E = P_d$. From Lemma 2, there is a projection F in $\mathcal{S}$ such that $\| E - F \| < \alpha(a)$ $(\leqq \frac{1}{30})$. Thus F is abelian, from Corollary D; and $\| C_E - C_F \| = \| P_d - C_F \| < 2\alpha(a)$, by Lemma 7. Since F is abelian in $\mathcal{S}$, $C_F \leqq Q_d$.

Symmetrically, there is a central subprojection P_0 of P_d such that $\| Q_d - P_0 \| < 2\alpha(a)$. Writing Q_1 for $Q_d - C_F$, we have

$$\| Q_1 - P_0Q_1 \| = \| (Q_d - P_0)Q_1 \| \leqq \| Q_d - P_0 \| < 2\alpha(a).$$

As $\| P_0(C_F - P_d)Q_1 \| = \| P_0Q_1 \| < 2\alpha(a)$; we have $\| Q_1 \| < 4\alpha(a) < 1$. Since Q_1 is a projection; $Q_1 = 0$ and $\| P_d - Q_d \| < 2\alpha(a)$.

LEMMA 11. *If $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras acting on a Hilbert space, $\| \mathcal{R} - \mathcal{S} \| < a(\leqq \frac{1}{750})$ and P_∞, Q_∞ are the maximal central projections in $\mathcal{R}$ and $\mathcal{S}$, respectively, such that $\mathcal{R}P_\infty$ and $\mathcal{S}Q_\infty$ are of type* III; *then $\| P_\infty - Q_\infty \| < \alpha(a)$.*

Proof. From Lemma 6, there is a central projection Q in $\mathcal{S}$ such that $\| P_\infty - Q \| < \alpha(a)$ $(\leqq \frac{1}{249})$. If F is a finite projection in $\mathcal{S}$ dominated by Q, from Lemma 9, there is a finite projection E in $\mathcal{R}P_\infty$ such that $\| E - F \| < \alpha(\frac{4}{747}) < \frac{12}{740}$ since $\| \mathcal{R}P_\infty - \mathcal{S}Q \| < a + \alpha(a) < \frac{4}{747}$. By definition of P_∞, $E = 0$ and $\| F \| < \frac{12}{740} < 1$. Thus $F = 0$. Since Q dominates no finite projection in $\mathcal{S}$, other than 0, $Q \leqq Q_\infty$.

Symmetrically, there is a central subprojection P of P_∞ in $\mathcal{R}$ such that $\| P - Q_\infty \| < \alpha(a)$. Writing Q_1 for $Q_\infty - Q$, $\| PQ_1 - Q_1 \| = \| (P - Q_\infty) Q_1 \| < \alpha(a)$; while $\| P(P_\infty - Q) Q_1 \| = \| PQ_1 \| < \alpha(a)$. Thus $\| Q_1 \| < 2\alpha(a) < 1$; and $Q_1 = 0$. Hence $\| P_\infty - Q_\infty \| < \alpha(a)$.

LEMMA 12. *If $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras acting on a Hilbert space,* $\| \mathcal{R} - \quad \| < a$ $(\leqq \frac{1}{2250})$, *and P_{c_1}, Q_{c_1}, P_{c_∞}, Q_{c_∞} are the maximal central projections in $\mathcal{R}$ and $\mathcal{S}$ such that $\mathcal{R}P_{c_1}$ and $\mathcal{S}Q_{c_1}$ are of type* II_1 *and* $\mathcal{R}P_{c_\infty}$ and $\mathcal{S}Q_{c_\infty}$ are of type II_∞; then $\|P_{c_1} - Q_{c_1}\| < \alpha(5a + 3\alpha(a))$ and $\| P_{c_\infty} - Q_{c_\infty} \| < 4a + 3\alpha(a) + \alpha(5a + 3\alpha(a))$.

Proof. If P and Q are the unit elements of $\mathcal{R}$ and $\mathcal{S}$, respectively, we can find A in $\mathcal{R}_1$ and B in $\mathcal{S}_1$ such that $\| A - Q \| < a$ and $\| P - B \| < a$. Thus

$$\| B - Q \| = \| BQ - Q \| \leqq \| BQ - BA \| + \| BA - PA \| + \| PA - A \| + \| A - Q \| < 3a,$$

and $\| P - Q \| < 4a$. Since $P_{c_1} + P_{c_\infty} = P - P_d - P_\infty$ and

$$Q_{c_1} + Q_{c_\infty} = Q - Q_d - Q_\infty; \| P_{c_1} + P_{c_\infty} - Q_{c_1} - Q_{c_\infty} \| < 4a + 3\alpha(a),$$

from Lemmas 10 and 11. Thus

$$\| \mathcal{R}(P_{c_1} + P_{c_\infty}) - \mathcal{S}(Q_{c_1} + Q_{c_\infty}) \| < 5a + 3\alpha(a) \quad (\leqq \tfrac{1}{10}).$$

From Lemma 6, there is a central subprojection Q_0 of $Q_{c_1} + Q_{c_\infty}$ such that $\| P_{c_1} - Q_0 \| < \alpha(5a + 3\alpha(a))$ $(< \alpha(\frac{1}{160}) < \frac{1}{52.8})$. Since P_{c_1} is finite relative to $\mathcal{R}$; Q_0 is finite relative to $\mathcal{S}$, by Lemma 9 (noting that $\frac{1}{2250} + \frac{2}{52.8} < \frac{1}{26}$). Thus $Q_0 \leqq Q_{c_1}$.

Symmetrically, there is a central subprojection P_0 of P_{c_1} in $\mathcal{R}$ such that $\| Q_{c_1} - P_0 \| < \alpha(5a + 3\alpha(a))$. Writing Q_1 for $Q_{c_1} - Q_0$, $\| P_0 Q_1 - Q_1 \| < \alpha(5a + 3\alpha(a))$ and $\| P_0(P_{c_1} - Q_0) Q_1 \| = \| P_0 Q_1 \| < \alpha(5a + 3\alpha(a))$. Thus $\| Q_1 \| < 1$ and $Q_1 = 0$. Hence $\| P_{c_1} - Q_{c_1} \| < \alpha(5\alpha + 3\alpha(a))$, and

$$\| P_{c_\infty} - Q_{c_\infty} \| < 4a + 3\alpha(a) + \alpha(5a + 3\alpha(a)).$$

Remark C. In the lemma which follows, we shall make use of the fact that if V and W are two finite-dimensional subspaces of a vector space X and V' is a complement for V in X, then $V' \cap W \neq (0)$ if $d(V) < d(W)$, where d is the dimension function on the set of subspaces of X. Working in the (finite-dimensional) subspace generated by V and W and noting that V' intersects this subspace in a complement, relative to it, for V; we may assume X is finite-dimensional. As $d(X) - d(V) = d(V')$,

$$d(W \wedge V') = d(X) - d(W \vee V') + d(W) - d(V) \geqq d(W) - d(V) \geqq 1.$$

LEMMA 13. *If $\mathcal{M}$ and $\mathcal{N}$ are finite factors of type I_m and I_n, respectively, in $\mathcal{B}(\mathcal{H})$, and $m < n$, then $\| \mathcal{M} - \mathcal{N} \| \nleqq \frac{1}{2}$.*

Proof. Let π be the projection of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$ given by

$$B \to \sum_{j,k=1}^{m} (Bx_j, x_k) E_{jk},$$

where $\{E_{jk}\}$ is a self-adjoint family of $m \times m$ matrix units for $\mathcal{M}$, x_1 is a unit vector in the range of E_{11}, and $x_k = E_{k1} x_1$. Then $\| \pi \| = 1$ and $\pi^{-1}(0)$ is an algebraic complement for $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$. It follows from the remark preceding this lemma that there is a B of norm 1 in $\mathcal{N}$ such that $\pi(B) = 0$. If A is in the unit ball of $\mathcal{M}$,

$$| \| A \| - 1 | = | \| A \| - \| B \| | \leqq \| A - B \|,$$

and

$$\| A \| = \| \pi(A - B) \| \leqq \| A - B \|;$$

so that

$$1 \leqq \| A \| + | 1 - \| A \| | \leqq 2 \| A - B \|;$$

and $\frac{1}{2} \leqq \| A - B \|$. Thus $\| \mathcal{M} - \mathcal{N} \| \nleqq \frac{1}{2}$.

LEMMA 14. *If $\mathcal{M}$ and $\mathcal{N}$ are infinite factors of types I_m and I_n, respectively, in $\mathcal{B}(\mathcal{H})$, and $\| \mathcal{M} - \mathcal{N} \| < \frac{1}{16}$ then $m = n$.*

Proof. We assume that $m < n$ and produce a contradiction. Let $\{E_j\}$ be a family of n orthogonal minimal projections in $\mathcal{N}$. From Lemma 3, we may choose n minimal projections $\{F_j\}$ in $\mathcal{M}$ such that $\| E_j - F_j \| < \frac{1}{4}$. As $\| E_j - E_k \| = 1$, for distinct j and k, $\| F_j - F_k \| \geqq \frac{1}{2}$. Now $\mathcal{M}$ is isomorphic to $\mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is m dimensional; and the isomorphism carries each F_j into a one-dimensional projection, G_j. Let x_j be a unit vector in the range of G_j. With x_0 an arbitrary unit vector in $\mathcal{K}$, since

$$\begin{aligned}(G_j - G_k)x_0 &= (x_0, x_j)x_j - (x_0, x_k)x_k \\ &= [(x_0, x_j) - (x_0, x_k)]x_j + (x_0, x_k)(x_j - x_k),\end{aligned}$$

$\| (G_j - G_k)x_0 \| \leqq 2 \| x_j - x_k \|$. Thus, when $j \neq k$,

$$\tfrac{1}{2} \leqq \| F_j - F_k \| = \| G_j - G_k \| \leqq 2 \| x_j - x_k \|.$$

As m is an infinite cardinal and $\mathcal{K}$ is m-dimensional, the finite, (complex-) rational linear combinations of an orthonormal basis for $\mathcal{K}$ forms a dense subset $\mathcal{S}$ of $\mathcal{K}$ having cardinality m. Since each vector in $\{x_j\}$ lies in the open ball of radius $\frac{1}{8}$ about some vector in $\mathcal{S}$ and there are n $(> m)$ elements in $\{x_j\}$; $\| x_j - x_k \| < \frac{1}{4}$ for some $j \neq k$, contradicting our earlier conclusion. Thus $n = m$.

LEMMA 15. *If $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras of type I acting on the Hilbert space $\mathcal{H}$ and P_n, Q_n are the respective central projections in $\mathcal{R}$ and $\mathcal{S}$ corresponding to their central portions of type I_n, $n = 1, 2, \cdots$, $\dim(\mathcal{H})$, then if $\| \mathcal{R} - \mathcal{S} \| < a$ $(\leqq \frac{1}{3660})$, $\| P_n - Q_n \| < \alpha(a)$. In particular, $P_n \neq 0$ if $Q_n \neq 0$.*

Proof. From Lemma 6, there is a central projection Q in $\mathcal{S}$ such that $\| P_n - Q \| < \alpha(a)$. We shall show that $Q = Q_n$. Note that $\| \mathcal{R}P_n - \mathcal{S}Q \| < a + \alpha(a)$ $(\leqq \frac{1}{914})$. If $\mathcal{S}Q$ is not of type I_n (that is, if $Q \nleqq Q_n$), there is a central projection Q_0 in $\mathcal{S}$ such that $Q_0 \leqq Q$ and $\mathcal{S}Q_0$ is of type I_m with $m \neq n$. Applying Lemma 6, again, there is a central subprojection P_0 of P_n in $\mathcal{R}$ such that $\| P_0 - Q_0 \| < \alpha(\frac{1}{914})$ $(< \frac{1}{304})$.

Writing $\mathcal{R}$ and $\mathcal{S}$ in place of $\mathcal{R}P_0$ and $\mathcal{S}Q_0$, we may assume that $\mathcal{R}$ is of type I_n, $\mathcal{S}$ is of type I_m, $n \neq m$, and $\| \mathcal{R} - \mathcal{S} \| < a + \alpha(\frac{1}{914}) < \frac{1}{280}$. We draw a contradiction from this; and conclude that $Q \leqq Q_n$. When we have done this, letting Q_0 be $Q_n - Q$, we will be able to conclude, from this same argument, that there is a central subprojection P_0 of P_n such that $\| P_0 - Q_0 \| < \alpha(a)$. But then, $\| P_0(P_n - Q) \| = \| P_0 - P_0Q \| < \alpha(a)$; while $\| P_0Q \| = \| (P_0 - Q_0)Q \| < \alpha(a)$ so that $\| P_0 \| < 2\alpha(a) < 1$. Thus $P_0 = 0$, and $\| Q_0 \| < \alpha(a) < 1$. Hence $Q_0 = 0$, $Q = Q_n$, and $\| P_n - Q_n \| < \alpha(a)$.

It remains to show that $m = n$ when $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras of types I_n and I_m, respectively, and $\| \mathcal{R} - \mathcal{S} \| < \frac{1}{280}$. We deal, first, with the case where the center of $\mathcal{R}$ has a joint (unit) eigenvector; and, for the sake of the general case (which will apply these considerations), we replace $\mathcal{R}$ and $\mathcal{S}$ by C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, acting on a Hilbert space $\mathcal{H}$. We assume that $\mathcal{A}$ is isomorphic to a von Neumann algebra of type I_n, $\mathcal{B}$ is isomorphic to a von Neumann algebra of type I_m, $\| \mathcal{A} - \mathcal{B} \| < \frac{1}{68}$, and that there is a unit vector x_0 in $\mathcal{H}$ such that each operator in the center $\mathcal{C}$ of $\mathcal{A}$ has x_0 as an eigenvector. By assumption $[\mathcal{C}x_0]$ is the one-dimensional projection with x_0 in its range—and, hence, minimal in $\mathcal{C}'$; so that $[\mathcal{C}'x_0]$ $(= P)$ is a minimal projection in $\mathcal{C}''N$, where N is the unit of $\mathcal{A}$. (We are assuming that $Nx_0 = x_0$, so that $P \leqq N$.) By assumption, $\mathcal{A}$ is generated by a C^*-subalgebra, $\mathcal{M}$, isomorphic to a type I_n factor and its center, $\mathcal{C}$. A self-adjoint family of matrix units for $\mathcal{M}$ generates a type I_n factor, $\mathcal{N}$, containing $\mathcal{M}$; and $\mathcal{N}'N$ contains $\mathcal{C}^-$. Now $N\mathcal{B}(\mathcal{H})N$ is unitarily equivalent to $\mathcal{N} \otimes \mathcal{N}'N$ acting on $N(\mathcal{H})$; so that the von Neumann algebra generated by $\mathcal{M}$ and $\mathcal{C}$ (that is, $\mathcal{A}^-$) is unitarily equivalent to $\mathcal{N} \otimes \mathcal{C}^-$, whose center is $I \otimes \mathcal{C}^-$. Thus $\mathcal{C}^-$ is the center of $\mathcal{A}^-$. Since P is minimal in $\mathcal{C}^-$ $(= \mathcal{C}''N)$, $\mathcal{A}^-P$ is a factor of type I_n. From Lemma 6 there is a central projection Q in $\mathcal{B}^-$ such that $\| P - Q \| < \alpha(\frac{1}{68}) < \frac{1}{22}$. Thus $\| \mathcal{A}^-P - \mathcal{B}^-Q \| < \frac{1}{68} + \frac{1}{22} < \frac{1}{16}$; and $\mathcal{B}^-Q$ is a factor (Corollary A). Since $\mathcal{B}^-Q$ is a factor of type I_m, Lemmas 13 and 14 tell us that $n = m$.

We may assume, henceforth, that neither the center of $\mathcal{R}$ nor that of $\mathcal{S}$ has a (joint) eigenvector (not in its null space). In particular, neither $\mathcal{R}$ nor $\mathcal{S}$ has projections minimal in its center; and neither has finite-dimensional projections in its center.

Since the set of cardinal numbers not exceeding the dimension of $\mathcal{H}$ is well-ordered, there is a non-zero projection P in the center $\mathcal{C}$ of $\mathcal{R}$ such that each non-zero, central projection in $\mathcal{R}$ has range with dimension not less than that of $P(\mathcal{H})$. From Lemma 6, there is a central projection Q in $\mathcal{S}$ such that $\| P - Q \| < \alpha(\frac{1}{280}) < \frac{1}{92}$. Now, $(I - Q) \wedge P = 0$ or else there is a unit vector x_0 such that $Qx_0 = 0$ and $Px_0 = x_0$; from which $1 = \| x_0 \| = \| (P - Q)x_0 \| < \frac{1}{90}$. The Kaplansky Formula [7; Theorem 5.4],

$$P - (I - Q) \wedge P \sim (I - Q) \vee P - (I - Q),$$

yields $P \sim (I - Q) \vee P - (I - Q) \leqq Q$; so that $\dim P(\mathcal{H})) \leqq \dim Q(\mathcal{H})$. Symmetrically, $\dim Q(\mathcal{H}) \leqq \dim P(\mathcal{H})$; so that the ranges of P and Q have the same dimension. If some non-zero central projection in $\mathcal{S}$ has range with dimension less than that of Q, the argument just given would produce a central projection in $\mathcal{R}$ with range of dimension less than that of P—contradicting the choice of P. It follows that Q is a central projection in $\mathcal{S}$ whose range has dimension the least cardinal number among the dimensions of the ranges of the non-zero central projections in $\mathcal{S}$.

Writing $\mathcal{R}$ and $\mathcal{S}$ in place of $\mathcal{R}P$ and $\mathcal{S}Q$, we may assume, henceforth, that $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras of types I_n and I_m, respectively, on the Hilbert space $\mathcal{H}$, that $\| \mathcal{R} - \mathcal{S} \| < \frac{1}{280} + \frac{1}{92} < \frac{1}{68}$, and that the dimension of the range of each non-zero central projection in $\mathcal{R}$ and $\mathcal{S}$ is equal to the dimension of $\mathcal{H}$ (working on the union of the ranges of P and Q, we may assume that $\mathcal{H}$ is this union).

Each (proper) norm-closed, two-sided ideal of $\mathcal{B}(\mathcal{H})$, other than the compact operators is characterized by an infinite cardinal number and consists of those operators on $\mathcal{H}$ whose ranges have dimension not exceeding that cardinal number. We prove that the intersection of the ideal $\mathcal{I}$ of compact operators on $\mathcal{H}$ with $\mathcal{R}$ and $\mathcal{S}$ is (0). If this is not the case, the intersection contains a non-zero projection E [2]. Since E is a compact operator, E has finite-dimensional range and dominates a minimal projection E_0. The central carrier of E_0 is, then, a minimal projection in the center—contradicting the present assumptions on $\mathcal{R}$ and $\mathcal{S}$. Thus $\mathcal{I} \cap \mathcal{R} = \mathcal{I} \cap \mathcal{S} = (0)$.

Suppose, now, that the ideal $\mathcal{I}$ is not the compact operators but consists of all operators on $\mathcal{H}$ whose range has dimension not exceeding the infinite cardinal a'. We show that, if A in $\mathcal{R}$ and T in $\mathcal{I}$ are such that $\| A - T \| \leqq 1$, then there is a T_0 in $\mathcal{I} \cap \mathcal{R}$ such that $\| A - T_0 \| \leqq 1$. If we have established this for positive A in $\mathcal{R}$, then, for arbitrary A in $\mathcal{R}$, we may write $A = VH$ where H $(= (A^*A)^{\frac{1}{2}})$ is a positive operator in $\mathcal{R}$ and V is a partial icometry in $\mathcal{R}$ with initial space the range of H. With T in $\mathcal{I}$ such that $\| A - T \| \leqq 1$, we have

$$\| H - T^*V \| = \| (A^* - T^*) V \| \leqq \| A^* - T^* \| \leqq 1.$$

As $T^*V \in \mathcal{I}$, there is, by assumption, a T_0 in $\mathcal{I} \cap \mathcal{R}$ such that $\| H - \mathrm{T}_0 \| \leqq 1$. Thus $\| A - VT_0 \| = \| V(H - T_0) \| \leqq \| H - T_0 \| \leqq 1$; and $VT_0 \in \mathcal{I} \cap \mathcal{R}$. We may assume, therefore, that A is positive. There is a projection E_ϵ (spectral for A) in $\mathcal{R}$, commuting with A, such that $(1 + \epsilon)E_\epsilon \leqq AE_\epsilon$ and

$\| A(I - E_\epsilon) \| \leqq 1 + \epsilon$ with $\epsilon > 0$. If the dimension of $E_\epsilon(\mathcal{H})$ exceeds a', then, as with our earlier use of the Kaplansky Formula, there is a unit vector x_0 in the range of E_ϵ orthogonal to range of a preassigned element of $\mathcal{I}$ (since each has range whose dimension does not exceed a'). If $\| A - T \| \leqq 1$ and $T \in \mathcal{I}$, we may assume that T is self-adjoint, since $T^* \in \mathcal{I}$ and $\| A - \frac{1}{2}(T + T^*) \| \leqq 1$. In this case, with x_0 chosen in $E_\epsilon(\mathcal{H})$ orthogonal to the range of T, $T^* x_0 = T x_0 = 0$; and

$$1 \geqq \| A - T \| \geqq ((A - T)x_0, x_0) = (Ax_0, x_0) = (AE_\epsilon x_0, x_0) \\ \geqq (1 + \epsilon)(E_\epsilon x_0, x_0) = 1 + \epsilon.$$

Hence $E_\epsilon(\mathcal{H})$ has dimension not exceeding a'. Writing E_n for E_ϵ, with $\epsilon = \frac{1}{n}$, we have $E_n \leqq E_m$ if $n \leqq m$. Since $E_n \in \mathcal{I} \cap \mathcal{R}$ for each n and $\bigvee_n E_n$ is a countable union of projections each of whose ranges has dimension not exceeding the infinite cardinal a', the same is true of $\bigvee_n E_n$ $(= E)$; so that $E \in \mathcal{I} \cap \mathcal{R}$. As $\| A(I - E_n) \| \leqq 1 + \frac{1}{n}$ and $(I - E_n)$ tends strongly to $I - E$, $\| A(I - E) \| = \| A - T_0 \| \leqq 1$, where $T_0 = AE \in \mathcal{I} \cap \mathcal{R}$.

We conclude from the foregoing that if ϕ is the natural mapping of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{I}$, then ϕ carries the unit ball of $\mathcal{R}$ and of $\mathcal{S}$ onto those of $\phi(\mathcal{R})$ and $\phi(\mathcal{S})$. To see this, we note that if $\mathcal{I}$ is the compact operators $\mathcal{I} \cap \mathcal{R} = \mathcal{I} \cap \mathcal{S} = (0)$; so that ϕ maps $\mathcal{R}$ and $\mathcal{S}$ isomorphically (hence, isometrically) onto $\phi(\mathcal{R})$ and $\phi(\mathcal{S})$. In case $\mathcal{I}$ is not the compact operators, if $\| A + \mathcal{I} \| < 1$, there is a T in $\mathcal{I}$, and hence in $\mathcal{I} \cap \mathcal{R}$ such that $\| A - T \| < 1$. But then $A - T$, in the unit ball of $\mathcal{R}$, is mapped by ϕ onto $A + \mathcal{I}$. This together with the fact that ϕ is norm-decreasing on $\mathcal{B}(\mathcal{H})$ allows us to conclude that $\| \phi(\mathcal{R}) - \phi(\mathcal{S}) \| \leqq \| \mathcal{R} - \mathcal{S} \|$.

We may also conclude that $\mathcal{I} \cap \mathcal{R} \neq (0)$ if and only if $\mathcal{I} \cap \mathcal{S} \neq (0)$ (when $\| \mathcal{R} - \mathcal{S} \| < \frac{1}{2}$); for if $T \in \mathcal{I} \cap \mathcal{R}$ and $\| T \| = 1$, there is a B in $\mathcal{S}$ such that $\| B - T \| < \frac{1}{2}$ and $\| B \| \leqq 1$. Thus $\frac{1}{2} < \| B \|$ and $\| 2B - 2T \| < 1$. From the foregoing, there is a T_0 in $\mathcal{I} \cap \mathcal{S}$ such that $\| 2B - 2T_0 \| \leqq 1$; so that $0 < \| B \| - \frac{1}{2} < \| T_0 \|$ and $T_0 \neq 0$.

If $\dim(\mathcal{H}) = b$ and $\mathcal{I}$ is the ideal characterized by the infinite cardinal a' $(< b)$, then $\mathcal{I} \cap \mathcal{R} = (0)$ if $n < b$. If $\mathcal{I} \cap \mathcal{R} \neq (0)$ it contains a projection and, hence, an abelian projection E in $\mathcal{R}$. Since $\mathcal{R}$ is of type I_n, the central carrier P of E is the sum n projections equivalent to E. Thus $\dim P(\mathcal{H}) = n \dim E(\mathcal{H})$. By arrangement, $\dim P(\mathcal{H}) = \dim(\mathcal{H}) = b$. Since n is less than the infinite cardinal b and $n \dim E(\mathcal{H}) = b$, $\dim E(\mathcal{H}) = b$—contradicting the choice of E in $\mathcal{I}$. Thus $\mathcal{I} \cap \mathcal{R} = (0)$ when $n < b$.

Since $n \neq m$, one of n or m differs from b, so that one of $\mathcal{R}$ or $\mathcal{S}$, and, from the foregoing, both $\mathcal{R}$ and $\mathcal{S}$ have intersection (0) with $\mathcal{J}$ for each ideal $\mathcal{J}$. To complete the argument, let ρ be an extension to $\mathcal{B}(\mathcal{H})$ of a pure state of the center of $\mathcal{R}$; and let ϕ be the representation of $\mathcal{B}(\mathcal{H})$ engendered by ρ on the Hilbert space $\mathcal{K}$, $\mathcal{J}$ be the kernel of ϕ and x_0 be a unit vector in $\mathcal{K}$ such that $\rho(T) = (\phi(T)x_0, x_0)$ for each T in $\mathcal{B}(\mathcal{H})$. Since ρ restricts to a pure state of $\mathcal{C}$ (the center of $\mathcal{R}$), and, in particular, $\rho(C^2) = \rho(C)^2$, for each C in $\mathcal{C}$; x_0 is a (joint) eigenvector for each $\phi(C)$ in $\phi(\mathcal{C})$, the center of $\phi(\mathcal{R})$. Having noted that the restrictions of ϕ to $\mathcal{R}$ and $\mathcal{S}$ are isomorphisms and that $\| \phi(\mathcal{R}) - \phi(\mathcal{S}) \| \leqq \| \mathcal{R} - \mathcal{S} \| < \frac{1}{68}$, the appropriate hypotheses for our earlier conclusion are fulfilled; and $n = m$.

The main result, which follows, is a consequence of Lemmas 10, 11, 12 and 15.

THEOREM A. *If $\mathcal{R}$ and $\mathcal{S}$ are von Neuman algebras acting on a Hilbert space $\mathcal{H}$, $\| \mathcal{R} - \mathcal{S} \| < a$ $(\leqq \frac{1}{26{,}000})$, P_d, P_{c_1}, P_{c_∞}, P_∞ and P_n, $n = 1, 2, \cdots, \dim(\mathcal{H})$ are the maximal central projections in $\mathcal{R}$ such that $\mathcal{R}P_d$, $\mathcal{R}P_{c_1}$, $\mathcal{R}P_{c_\infty}$, $\mathcal{R}P_\infty$* and *$\mathcal{R}P_n$ are of types I, II_1, II_∞, III and I_n, respectively, and Q_d, Q_{c_1}, Q_{c_∞} Q_∞, and Q_n, $n = 1, 2, \cdots, \dim(\mathcal{H})$ are the corresponding central projections for $\mathcal{S}$; then $\| P_d - Q_d \| < 2\alpha(a)$,*

$$\| P_{c_1} - Q_{c_1} \| < \alpha(5a + 3\alpha(a)),$$
$$\| P_{c_\infty} - Q_{c_\infty} \| < 4a + 3\alpha(a) + \alpha(5a + 3\alpha(a)),$$

$\| P_\infty - Q_\infty \| < \alpha(a)$ and $\| P_n - Q_n \| < \alpha(a + 2\alpha(a))$. In particular, the same types occur in the type decomposition of $\mathcal{R}$ and $\mathcal{S}$.

If we resctrict attention to the most basic situation, that of factors containing I on a separable Hilbert space the conclusions simplify considerably. We state these in:

THEOREM B. *If $\mathcal{M}$ is a factor and $\mathcal{N}$ is a von Neumann algebra both containing the identity operator I and both acting on the separable Hilbert space $\mathcal{H}$, then $\mathcal{N}$ is a factor of the same type as $\mathcal{M}$ if $\| \mathcal{M} - \mathcal{N} \| < \frac{1}{8}$.*

Proof. As in the second paragraph of the proof of Lemma 3, if F is a central projection in $\mathcal{N}$, there is a projection E in $\mathcal{M}$ such that $\| E - F \| < \alpha(\frac{1}{16}) < \frac{1}{4}$. From Corollary B, E is central in $\mathcal{M}$, since $\| E - F \| < \frac{1}{4} < \frac{1}{2} - \frac{1}{8}$. Since $\mathcal{M}$ is a factor, $E = 0$ or $E = I$; so that $F = 0$ or $F = I$ (from $\| E - F \| < \frac{1}{4}$). Thus $\mathcal{N}$ is a factor.

If $\mathcal{M}$ is of type I_m and E is a minimal projection in $\mathcal{M}$; then, from the proof of Lemma 3, there is a minimal projection F in $\mathcal{N}$ (such that $\| E - F \| < \frac{1}{4}$). Thus $\mathcal{N}$ is a factor of type I. From Lemmas 13 and 14, $\mathcal{N}$ is of type I_m. (If m is an infinite cardinal and Lemma 14 is applied, the proof of Lemma 3 allows us to locate minimal projections F_j such that $\| E_j - F_j \| < \frac{1}{4}$, since I is in both $\mathcal{M}$ and $\mathcal{N}$. The proof of Lemma 14, then, proceeds as given.)

If I is equivalent to a proper subprojection F in $\mathcal{M}$ (so that $\mathcal{M}$ is infinite), there is a projection N in $\mathcal{N}$ such that $\| F - N \| < \frac{1}{4}$. Replacing E and M by I in the proof of Lemma 8, we find that $\| B - V \| < \frac{1}{4}$, $\| B^*B - I \| < \frac{1}{2}$, and $\| BB^* - N \| < 1$; so that, as in that argument, I is equivalent to N in $\mathcal{N}$. Since $\| I - F - (I - N) \| < \frac{1}{4}$ and $\| I - F \| = 1$; $N \neq I$. Thus $\mathcal{N}$ is infinite.

If $\mathcal{M}$ is of type II_1 $\mathcal{N}$ is not of type I, from the second paragraph of this proof, yet $\mathcal{N}$ is finite. Thus $\mathcal{N}$ is of type II_1.

If $\mathcal{M}$ is of type III and F is a non-zero projection in $\mathcal{N}$, there is a projection E in $\mathcal{M}$ such that $\| E - F \| < \frac{1}{4}$. Since $\mathcal{H}$ is separable and $\mathcal{M}$ is a factor of type III; E is equivalent to I in $\mathcal{M}$. (Note that $E \neq 0$ since $F \neq 0$ and $\| E - F \| < \frac{1}{4}$.) As before, F is equivalent to I in $\mathcal{N}$. Thus $\mathcal{N}$ is a factor of type III.

Finally, if $\mathcal{M}$ is a factor of type II_∞ then, from the preceding, $\mathcal{N}$ is not of type I, II_1 or III; so that $\mathcal{N}$ is a factor of type II_∞.

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